Alternator Coins

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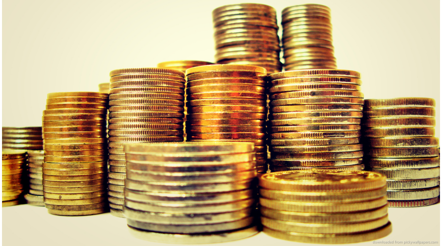
PRIMES CONFERENCE, May 21, 2016

Original Coin Problem



You are given N coins that look identical, but one of them is fake and is lighter than other coins. You have a balance scale that you can use to help find the fake coin. What is the smallest number of weighings that guarantees finding the fake coin?

Example

This is a coin problem that first appeared in 1945. Since then, there were many generalizations of this puzzle.

Try this problem: What is the smallest number of weighings that guarantees finding the fake coin from a group of eight coins?

Answer

Answer: 2

Original Coin Problem — Solution to All Cases

For a case with N coins, the number of weighing will be $\lceil \log_3 N \rceil$.

Alternator Coin Idea

Alternator Coin: A coin that starts out randomly: fake or real, and then after each weighing that it participates in, it switches state.

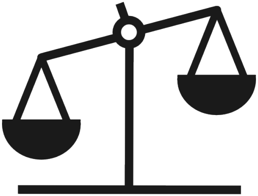
Alternator Coin — States

f-state — The alternator coin will act as a fake coin in its next weighing.

r-state — The alternator coin will act as a real coin in its next weighing.

Solutions

N is the total number of coins.
$f(N)$ — The smallest number of weighings to find the alternator if the alternator coin is currently in the f-state.
$r(N)$ — The smallest number of weighings to find the alternator if the alternator coin is currently in the r-state.
$a(N)$ — smallest number of weighings to find the alternator if the state of the alternator is unknown.



Trivial Bounds

There are trivial lower and upper bounds:

- **Lower bound:** the alternator is worse than the fake coin.
- **Upper bound:** the alternator is better than two times the weighings needed for one fake coin.

Example: 3 coins

What are $r(N)$ and $f(N)$ for 3 coins?

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$\mathrm{r}(N) = 2$
$\mathrm{f}(N) = 1$

Jacobsthal Numbers

Sequence J_n: 0, 1, 1, 3, 5, 11, 21, 43...
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$$J_n = (2^n - (-1)^n)/3.$$

Observations

We made the following observations:

- The number of weighings necessary increases by one after the number of coins reaches the next Jacobsthal number.
- $f(N)$ is always equal to $r(N) - 1$.
- $r(N) = a(N)$.

Ideal Strategy

There are 11 coins, one of which is an alternator coin. How many weighings on a two pan balance will it take to find the alternator coin?

Strategy: f-state

$11 = J_5$
$J_5 = J_4 + 2J_3$: $11 = 5 + 2 \cdot 3$.
We compare 3 coins versus 3 coins.

- If unbalances: $r(3) = 2$.
- If balances: $f(5) = 2$.

Thus, $f(11) = 3$.

Strategy: r-state

- Even number of coins: put all of them on the scale: $r(2k) = f(2k) + 1$.
- Odd number of coins: put one aside. Later if everything balances, then this is the alternator: $r(2k+1) = f(2k) + 1$.

Optimum I

- E: equal
- L: left pan is heavier
- R: right pan is heavier

Every unique string points to a different coin.

Optimum II

Property: L or R must be followed by E.
The number of such strings of length n is J_{n+2}:

- Length 0: one string: only empty string.
- Length 1: three strings, E,L,R.

The number of such strings, $s(n)$:
$s(n) = s(n-1) + 2s(n-2)$,
For the r-state the string has to start with E, so the number of such strings of length n is J_{n+1}.

Results

Theorem

For the f state, the number of coins N we can process in w weighings is $J_{w+1} < N \leq J_{w+2}$. For the r state, the number of coins N we can process in w weighings is $J_w < N \leq J_{w+1}$.

Theorem

For the a-state and r-state, the number of coins N we can process in w weighings is $J_w < N \leq J_{w+1}$.

Corollary

$a(N) = r(N) = f(N) + 1.$

Acknowledgements

We would like to thank the PRIMES STEP program for the opportunity to do this research. In addition, we are grateful to PRIMES STEP Director, Doctor Slava Gerovitch, for his help and support.